

Relation Between Quantum Speed Limits And Metrics On $U(n)$

Kai-Yan Lee* and H. F. Chau†

*Department of Physics and Center of Computational and Theoretical Physics
University of Hong Kong, Pokfulam Road, Hong Kong*

(Dated: February 10, 2012)

Recently, Chau [Quant. Inform. & Comp. **11**, 721 (2011)] found a family of metrics and pseudo-metrics on n -dimensional unitary operators that can be interpreted as the minimum resources (given by certain tight quantum speed limit bounds) needed to transform one unitary operator to another. This result is closely related to the weighted ℓ^1 -norm on \mathbb{R}^n . Here we generalize this finding by showing that every weighted ℓ^p -norm on \mathbb{R}^n with $1 \leq p \leq \pi/2$ induces a metric and a pseudo-metric on n -dimensional unitary operators with quantum information-theoretic meanings related to certain tight quantum speed limit bounds. Besides, we investigate how far the correspondence between the existence of metrics and pseudo-metrics of this type and the quantum speed limits can go.

PACS numbers: 03.65.Aa, 03.67.-a, 89.70.Eg

I. INTRODUCTION

Distinguishing two quantum operations and characterizing the resources needed to carry out a quantum operation are two meaningful problems in quantum information science. Various authors have studied the former problem. For instance, the problem of unambiguously distinguishing two quantum operators have been extensively studied [1–3]. Whereas one way to attack the latter problem is through the so-called quantum speed limits (QSLs) which put lower bounds on the evolution time needed to perform a unitary operation [4–7].

Note that a few QSLs have geometric meanings. For instance, the well-known time-energy uncertainty relation [8] comes from the Bures metric on the group of unitary operators [9]. And the recently discovered families of metrics and pseudo-metrics on the group of n -dimensional unitary matrices $U(n)$ by Chau [10] are closely related to a QSL involving the average absolute deviation from the median energy. Actually, for any $U, V \in U(n)$, these metrics and pseudo-metrics can be written as certain weighted sums of the absolute value of the argument of eigenvalue of the unitary matrix UV^{-1} ; hence, they are related to certain weighted average of the absolute value of the energy eigenvalues of the Hermitian operator generating UV^{-1} . Lately, Chau *et al.* went further to show that these families of metrics and pseudo-metrics can be induced by the symmetric weighted ℓ^1 -norm on \mathbb{R}^n [11]. (We will define symmetric weighted ℓ^p -norm for $p \geq 1$ in Sec. II.) More importantly, they [11] interpreted these metrics and pseudo-metrics as the consequence of certain “reasonable” cost functions to implement a unitary operation given by the tight QSL bound reported in Ref. [7]. (We will clarify what we mean by a “reasonable” cost function in Sec. IV.)

It is instructive to study how close the relation between the implementation cost in terms of, say, certain QSLs and the existence of metrics or pseudo-metrics on $U(n)$ induced by such cost functions. Here we extend the findings by Chau and his co-workers [7, 10, 11] by proving the following results. First, for $p \geq 1$, there are metrics and pseudo-metrics on $U(n)$ that are functions of $|\theta_j|^p$'s where $e^{i\theta_j}$'s are the eigenvalues of the unitary matrix UV^{-1} with $\theta_j \in (-\pi, \pi]$ for all j . In fact, these metrics and pseudo-metrics are induced by certain symmetric weighted ℓ^p -norms on \mathbb{R}^n . Second, for every $p > 0$, there are two QSL bounds. One involves $\langle |E|^p \rangle^{1/p}$, the p th root of the p th moment of the absolute value of energy of the system; and the other involves $\mathcal{D}_p E$, which is an optimized version of $\langle |E|^p \rangle^{1/p}$ by exploiting the freedom of choosing the reference energy level. Most importantly, these bounds are tight for $p \leq \pi/2$. Thus, for $1 \leq p \leq \pi/2$, the metrics and pseudo-metrics reported here can be interpreted as the minimum resources needed (through the tight QSL bounds involving $\langle |E|^p \rangle$ and $\mathcal{D}_p E$ respectively) to convert one unitary operator to another. Nevertheless, our findings imply that for $p < 1$ or $p \geq \pi/2$, this close relation between the metric / pseudo-metric and the QSL breaks down because either the induced metric / pseudo-metric no longer exists or the QSL is no longer a tight bound. This work is a refinement and improvement of the research reported in the M.Phil. thesis of the first author [12].

II. METRICS AND PSEUDO-METRICS INDUCED BY WEIGHTED ℓ^p -NORMS

We say that a function $g: \mathbb{R}^n \rightarrow [0, \infty)$ a **symmetric norm** on \mathbb{R}^n if g is a norm on \mathbb{R}^n satisfying $g(\mathbf{v}) = g(\mathbf{v}P)$ for any $\mathbf{v} \in \mathbb{R}^n$, and any permutation matrix or diagonal orthogonal matrix P .

Recall that for any fixed $p \geq 1$, a weighted ℓ^p -seminorm on \mathbb{R}^n is a function $h: \mathbb{R}^n \rightarrow [0, \infty)$ in the form $h(\mathbf{v}) \equiv h(v_1, v_2, \dots, v_n) = (\sum_{j=1}^n \mu_j |v_j|^p)^{1/p}$ for some $\mu_j \geq 0$ for all j . Surely, a weighted ℓ^p -seminorm is indeed a

* Present address: Department of Astronomy and Oskar Klein Centre for Cosmoparticle Physics, Stockholm University, AlbaNova, SE-10691 Stockholm, Sweden; lee.kai_yan@astro.su.se

† Corresponding author; hfchau@hku.hk

seminorm on \mathbb{R}^n .

For any weighted ℓ^p -seminorm h , we may define

$$\begin{aligned} g(\mathbf{v}) &= \max_P h(v_{P(1)}, v_{P(2)}, \dots, v_{P(n)}) \\ &= \left[\sum_{j=1}^n \mu_j^\downarrow \left(|v|_j^\downarrow \right)^p \right]^{\frac{1}{p}}, \end{aligned} \quad (1)$$

where the maximum is over all permutations P of $\{1, 2, \dots, n\}$. Besides, μ_j^\downarrow and $|v|_j^\downarrow$ denote the j th largest number in the sequences $(\mu_1, \mu_2, \dots, \mu_n)$ and $(|v_1|, |v_2|, \dots, |v_n|)$, respectively. It is straightforward to check that g is a symmetric norm on \mathbb{R}^n provided that not all μ_j 's are 0; and we call this particular type of symmetric norm the **symmetric weighted ℓ^p -norm**. (By taking the limit $p \rightarrow +\infty$, we have a symmetric weighted ℓ^∞ -norm. This symmetric weighted ℓ^∞ -norm is a special case of symmetric weighted ℓ^1 -norm in which all but one of the weights μ_j are 0. So, we will not pay particular

attention to symmetric weighted ℓ^∞ -norm any further.)

For any symmetric weighted ℓ^p -norm on \mathbb{R}^n , we may apply the following result by Chau *et al.* in Ref. [11] to induce a metric and a pseudo-metric on $U(n)$:

Proposition 1 (Chau *et al.*). *For any given symmetric norm $g: \mathbb{R}^n \rightarrow [0, \infty)$, we may define a metric d_g and a pseudo-metric d_g^∇ on $U(n)$ by*

$$d_g(U, V) = g(|\theta|_1^\downarrow(UV^{-1}), \dots, |\theta|_n^\downarrow(UV^{-1})) \quad (2)$$

and

$$d_g^\nabla(U, V) = \min_{x \in \mathbb{R}} g(|\theta|_1^\downarrow(e^{ix}UV^{-1}), \dots, |\theta|_n^\downarrow(e^{ix}UV^{-1})). \quad (3)$$

Here $|\theta|_j^\downarrow(UV^{-1})$ denotes the j th largest number in the sequence $(|\theta_1|, |\theta_2|, \dots, |\theta_n|)$ with $e^{i\theta_j}$'s being the eigenvalues of UV^{-1} obeying $-\pi < \theta_j \leq \pi$.

Corollary 1. *Suppose $p \geq 1$. Then,*

$$d_{p, \vec{\mu}}(U, V) = \left\{ \sum_{j=1}^n \mu_j^\downarrow \left[|\theta|_j^\downarrow(UV^{-1}) \right]^p \right\}^{\frac{1}{p}} = \left(\sum_{j=1}^n \mu_j \right) \min_{Ht: \exp(-iHt/\hbar) = UV^{-1}} \max_{|\phi\rangle \in C(H, \vec{\mu})} [\langle |E|^p \rangle(H, |\phi\rangle)]^{\frac{1}{p}} t \quad (4)$$

and

$$\begin{aligned} d_{p, \vec{\mu}}^\nabla(U, V) &= \min_{x \in \mathbb{R}} d_{p, \vec{\mu}}(e^{ix}U, V) = \min_{x \in \mathbb{R}} \left\{ \sum_{j=1}^n \mu_j^\downarrow \left[|\theta|_j^\downarrow(e^{ix}UV^{-1}) \right]^p \right\}^{\frac{1}{p}} \\ &= \min_{x \in \mathbb{R}} \left(\sum_{j=1}^n \mu_j \right) \min_{Ht: \exp(-iHt/\hbar) = e^{ix}UV^{-1}} \max_{|\phi\rangle \in C(H, \vec{\mu})} \mathcal{D}_p E(H, |\phi\rangle) t \end{aligned} \quad (5)$$

are metric and pseudo-metric on $U(n)$, respectively. Here $C(H, \vec{\mu})$ is the set of all (normalized) state kets in the form $\sum_{j=1}^n \alpha_j |E_{P(j)}\rangle$, $|E_j\rangle$ is the energy eigenstate of H with energy E_j , $|\alpha_j|^2 = \mu_j / \sum_k \mu_k$, and P is a permutation of $\{1, 2, \dots, n\}$. Also,

$$\langle |E|^p \rangle \equiv \langle |E|^p \rangle(H, |\phi\rangle) = \text{Tr}(|H|^p |\phi\rangle \langle \phi|) = \langle \phi | |H|^p | \phi \rangle \quad (6)$$

is the p th moment of the absolute value of energy of the system and

$$\mathcal{D}_p E \equiv \mathcal{D}_p E(H, |\phi\rangle) = \min_{x \in \mathbb{R}} [\langle |E|^p \rangle(H - xI, |\phi\rangle)]^{\frac{1}{p}} \quad (7)$$

is the p th root of the p th moment of the absolute value of energy of the system minimized over the reference energy level.

Proof. Applying Proposition 1 to the symmetric weighted ℓ^p -norm in Eq. (1) gives the first equality in Eq. (4) as well as the first line of Eq. (5). More importantly, it

implies that $d_{p, \vec{\mu}}$ and $d_{p, \vec{\mu}}^\nabla$ are metric and pseudo-metric, respectively.

To show the second equality in Eq. (4), we adopt the strategy used in the proof of Theorem 1 in Ref. [10]. Note that $\langle |E|^p \rangle(H, \sum_j \alpha_j |E_j\rangle) = \sum_j |\alpha_j|^2 |E_j|^p$. So, the R.H.S. of Eq. (4) becomes $\min [\sum_j \mu_j^\downarrow (|E|_j^\downarrow)^p]^{1/p}$, where $|E|_j^\downarrow$ denotes the j th largest element in the sequence $(|E_1|, |E_2|, \dots, |E_n|)$. Among those Ht 's that satisfy $\exp(-iHt/\hbar) = UV^{-1}$, the one that minimizes $[\sum_j \mu_j^\downarrow (|E|_j^\downarrow)^p]^{1/p}$ can always be picked in such a way that its eigenvalues all lie in $(-\pi, \pi]$ [10]. Hence, the R.H.S. of Eq. (4) is reduced to $\{\sum_j \mu_j^\downarrow [|\theta|_j^\downarrow(UV^{-1})]^p\}^{1/p}$.

We omit the proof of the last line of Eq. (5) for it is essentially the same as that of the second inequality in Eq. (4). \square

Remark 1. *From the above proof, we know that Eqs. (4) and (5) hold irrespective of whether $d_{p, \vec{\mu}}$ is a metric or not. We note further that the special cases of $d_{1, \vec{\mu}}$ and*

$d_{1,\vec{\mu}}^\nabla$ are the metric and pseudo-metric reported originally by Chau in Ref. [10]. Moreover, it is possible to use an elementary method involving Minkowski inequality to show that $d_{p,\vec{\mu}}^\nabla$ and $d_{p,\vec{\mu}}^\nabla$ are metric and pseudo-metric, respectively. Details can be found in the master thesis of the first author [12].

III. QUANTUM SPEED LIMITS VIA $\langle |E|^p \rangle^{\frac{1}{p}}$ OR $\mathcal{D}_p E$

We extend the proof concept used by Chau in Ref. [7] to find these QSLs. And we begin with the following lemma.

Lemma 1. *Let $0 < p \leq 2$. Then $A_p \equiv \sup \{(1 - \cos x)/x^p : x > 0\}$ exists and is equal to $\max_{x \in [0, \pi]} (1 - \cos x)/x^p > 0$. In fact the maximum is attained by a unique $x_c \in [0, \pi]$. And this unique x_c is a decreasing function of p with $x_c > 0$ for $p < 2$ and $x_c = 0$ when $p = 2$. Thus,*

$$\cos x \geq 1 - A_p |x|^p \quad (8)$$

for all $x \in \mathbb{R}$ with equality hold if and only if $x = 0, \pm x_c$.

Let us talk about the geometric meaning of the lemma before proving it. The lemma means that the curve $y = \cos x$ is always above the curve $y = 1 - A|x|^p$ provided that A is a sufficiently large positive number. Besides, A_p is the least possible value of A for this to happen.

Proof. Let $f(x) = (1 - \cos x)/x^p$ for $x > 0$ and $f(0) = \lim_{x \rightarrow 0^+} (1 - \cos x)/x^p$. Since $p \leq 2$, f is well-defined and continuous in $[0, \infty)$. Moreover, $f(x) > (1 - \cos x)/(x + 2\pi)^p = f(x + 2\pi)$ for all $x > 0$ and $f(x) > f(2\pi - x)$ for $0 \leq x < \pi$ because $p > 0$. Hence, $A_p = \max \{f(x) : x \in [0, \pi]\} \geq f(\pi) > 0$ and the maximum is attained by a certain $x_c \in [0, \pi]$.

Surely, $df/dx|_{x=x_c} = 0$ which can be simplified to

$$p \tan \frac{x_c}{2} = x_c. \quad (9)$$

Note that the slope of the curve $y = \tan(x/2)$ is strictly increasing for $x \in [0, \pi)$ and is equal to $1/2$ at $x = 0$. Hence, for $p = 2$, the only solution of Eq. (9) in the domain $[0, \pi]$ is $x_c = 0$. Whereas for $p \in (0, 2)$, $f(0) = 0$.

So, $x_c > 0$ as $A_p > 0$. Now consider the continuous function $\tilde{f}(x) = \tan(x/2)/x$ in $(0, \pi)$ with $\tilde{f}[(0, \pi)] = (1/2, \infty)$. This function is strictly increasing in $(0, \pi)$ for $d\tilde{f}(x)/dx = (x - \sin x) \sec^2(x/2)/2x^2 > 0$ for $0 < x < \pi$. Thus, the equation $\tilde{f}(x) = 1/p$ has a unique solution in the domain $(0, \pi)$ for all $p \in (0, 2)$. Clearly, this unique solution is the required x_c that maximizes $f(x)$. More importantly, since \tilde{f} is strictly increasing in $(0, \pi)$, x_c decreases as p increases.

Since the L.H.S. and R.H.S. of Eq. (8) are even functions, we only need to prove its validity for $x \geq 0$. The case of $x > 0$ is a consequence of $A_p \geq (1 - \cos x)/x^p$ for all $x > 0$; whereas the case of $x = 0$ is trivial. Finally, the if and only if condition for Eq. (8) to be an equality follows from the fact that $f(x)$ is maximized by a unique $x = x_c$. \square

Corollary 2. *Let $p > 0$, and $H = \sum_{j=1}^n E_j |E_j\rangle \langle E_j|$ be a time-independent Hamiltonian acting on an n -dimensional Hilbert space. Then the time τ needed to evolve a pure state $|\Phi(0)\rangle$ to $|\Phi(\tau)\rangle$ under the action of H is lower-bounded by*

$$\tau \geq \tau_{c1} \equiv \hbar \left(\frac{1 - \sqrt{\epsilon}}{A_p \langle |E|^p \rangle} \right)^{\frac{1}{p}}. \quad (10)$$

Here $\epsilon = F(|\Phi(0)\rangle, |\Phi(\tau)\rangle) \equiv |\langle \Phi(0) | \Phi(\tau) \rangle|^2$ is the fidelity between the two states, and $\langle |E|^p \rangle$ is the p th moment of the absolute value of energy of the system defined by Eq. (6). Also, A_p is given by Lemma 1 if $p \leq 2$ and A_p is defined to be A_2 otherwise. Actually, we can slightly optimize the bound in Eq. (10) to

$$\tau \geq \tau_{c2} \equiv \frac{\hbar}{\mathcal{D}_p E} \left(\frac{1 - \sqrt{\epsilon}}{A_p} \right)^{\frac{1}{p}}, \quad (11)$$

where $\mathcal{D}_p E$ is the p th root of the p th moment of the absolute value of energy of the system minimized over the reference energy level as defined by Eq. (7). More importantly, these two bounds are tight for all $\epsilon \in [0, 1]$ if $p \leq \pi/2$.

Proof. We first prove Eq. (10) for the case of $p \leq 2$. The initial quantum state $|\Phi(0)\rangle$ can be written in the form $\sum_{j=1}^n \alpha_j |E_j\rangle$ with $\sum_{j=1}^n |\alpha_j|^2 = 1$. From Lemma 1, the time $\tau > 0$ needed to evolve to a state $|\Phi(\tau)\rangle$ with $F(|\Phi(0)\rangle, |\Phi(\tau)\rangle) = \epsilon$ must obey

$$\begin{aligned} \sqrt{\epsilon} &= |\langle \Phi(0) | \Phi(\tau) \rangle| = \left| \sum_{j=1}^n |\alpha_j|^2 e^{-iE_j\tau/\hbar} \right| \geq \left| \sum_{j=1}^n |\alpha_j|^2 \cos \left(\frac{E_j\tau}{\hbar} \right) \right| \geq \sum_{j=1}^n |\alpha_j|^2 \cos \left(\frac{E_j\tau}{\hbar} \right) \\ &\geq \sum_{j=1}^n |\alpha_j|^2 \left(1 - A_p \left| \frac{E_j\tau}{\hbar} \right|^p \right) = 1 - \frac{A_p \tau^p}{\hbar^p} \sum_{j=1}^n |\alpha_j|^2 |E_j|^p = 1 - \frac{A_p \tau^p \langle |E|^p \rangle}{\hbar^p}. \end{aligned} \quad (12)$$

Hence, the QSL in Eq. (10) is valid whenever $0 < p \leq 2$.

To prove the validity of this QSL for the case of $p > 2$, we only need to combine Eq. (10) for $p = 2$ and $\langle |E|^p \rangle^{1/p} \geq \langle |E|^2 \rangle^{1/2}$ for all $p > 2$, which is a special case of the Lyapunov's inequality [13].

Now have just established the truth of the QSL in Eq. (10). The other QSL given by Eq. (11) follows by the fact that the reference energy level has no physical meaning. So from Eq. (10), we can obtain a more “optimized” bound by varying the reference energy level x so as to minimize $\langle |E|^p \rangle (H - xI, |\Phi(0)\rangle)$ [7]. Therefore, Eq. (11) follows from Eqs. (7) and (10).

To show that the two QSLs are tight bounds for $p \leq \pi/2$, we only need to give an example of initial state that saturates the bound for all fidelity $\epsilon \in [0, 1]$. And since the bound in Eq. (11) is in general more stringent than the bound in Eq. (10), we only need to show that the example we give saturates the former bound. Note further that there is no need to check for the case of $\epsilon = 1$ because the QSLs reduce to $\tau \geq 0$ which is trivially true. Now we claim that the following state saturates the QSL stated in Eq. (10):

$$|\Phi(0)\rangle = \sqrt{1-\beta}|0\rangle + \sqrt{\frac{\beta}{2}}(|E\rangle + |-E\rangle) \quad (13)$$

where $E > 0$ and $\beta = (1 - \sqrt{\epsilon})/A_p x_c^p$ with x_c being the maximum point defined in Lemma 1 so that $\cos x_c = 1 - A_p x_c^p$. (Note that β is well-defined as Lemma 1 demands $A_p > 0$.) Since $p \leq \pi/2 < 2$, Lemma 1 implies $x_c > 0$. Thus, $\beta \geq 0$ for all $\epsilon \in [0, 1]$. As x_c is a decreasing function of p obeying Eq. (9), $A_p x_c^p = 1 - \cos x_c \geq 1$ (and hence $\beta \leq 1$ for all $\epsilon \in [0, 1]$) whenever $0 < p \leq p_c$ where p_c is the critical value of p in $(0, 2]$ such that $\cos x_c = 0$ and hence $x_c = \pi/2$. From Eq. (9), $p_c = \pi/2$. In conclusion, Eq. (13) is a valid quantum state if $0 < p \leq \pi/2$. Since $A_p, x_c > 0$ and $\epsilon < 1$, it is easy to check that for this particular quantum state, $\langle |E|^p \rangle = \beta E^p$ and $\langle \Phi(0) | \Phi(\tau_c) \rangle = 1 - \beta + \beta \cos x_c = \sqrt{\epsilon}$. So Eq. (10) is tight for all $\epsilon \in [0, 1]$ provided that $p \in (0, \pi/2]$. \square

Remark 2. From the above proof, for $\pi/2 < p < 2$, Eqs. (10) and (11) are tight for some but not all $\epsilon \in [0, 1]$ because Eq. (13) is a valid quantum state for ϵ sufficiently close to 1. Note further that for $p = 1$, Corollary 2 reduces to an earlier result obtained by Chau in Ref. [7]. Actually, the QSLs reported here also apply to the case of mixed state through the use of the purification argument by Giovannetti et al. in Ref. [6]. Hence, these two QSLs can be regarded as fundamental limit on the minimum time needed to evolve a density matrix or alternatively as a fundamental limit on the maximum possible information processing rate of a system [4–6].

IV. CONNECTION BETWEEN THE METRICS, PSEUDO-METRICS AND THE QUANTUM SPEED LIMIT BOUNDS

By comparing $d_{p,\bar{\mu}}$ in Eqs. (4) and $d_{p,\bar{\mu}}^\nabla$ in Eq. (5) of Corollary 1 with the QSLs involving $\langle |E|^p \rangle$ or $\mathcal{D}_p E$ in Corollary 2, we may interpret $d_{p,\bar{\mu}}$ and $d_{p,\bar{\mu}}^\nabla$ as cost functions describing the minimum amount of resources needed to convert U from V . In the first case, the resources refer to the product of evolution time τ and the p th root of the p th moment of absolute value of energy of the system $\langle |E|^p \rangle^{1/p}$ required to carry out the conversion. And in the second case, the resources refer to the product of τ and $\mathcal{D}_p E$ (which is an “optimized” version of $\langle |E|^p \rangle^{1/p}$) [10].

Three remarks are in place. First, since this connection is done via QSL bounds, it works best when the bounds are tight for all ϵ . For otherwise, the cost functions always overestimate the actual resources required. So, from Corollary 2, this connection begins to lose its significance when $p > \pi/2$.

Second, from Remark 1, we know that this interpretation works whenever $p > 0$ — that is, even in the case when $d_{p,\bar{\mu}}$ is not a metric. However, Chau *et al.* [11] argued that any “reasonable” cost functions $d_{p,\bar{\mu}}$ and $d_{p,\bar{\mu}}^\nabla$ should be a metric and a pseudo-metric on $U(n)$, respectively. Part of the reasons is that one way to transform V to U is first transforming V to W and then from W to U . So, $d_{p,\bar{\mu}}$ and $d_{p,\bar{\mu}}^\nabla$ must satisfy the triangle inequality if the cost of transformation is additive — a rather modest additional requirement indeed. In this regard, the cost functions $d_{p,\bar{\mu}}$ and $d_{p,\bar{\mu}}^\nabla$ in Eqs. (4) and (5) are “reasonable” provided that $p \geq 1$.

Finally, since the overall phase of a unitary operator has no physical significance, the cost function $d_{p,\bar{\mu}}^\nabla(U, V)$ is more meaningful than $d_{p,\bar{\mu}}(U, V)$ in characterizing the resources needed to transform V to U . Nonetheless, $d_{p,\bar{\mu}}$ is important in its own right for it gives rise to a characterization on the degree of non-commutativity between two unitary operators U and V through $d_{p,\bar{\mu}}(UV, VU)$ [10].

To summarize, we have shown that any symmetric weighted ℓ^p -norm on \mathbb{R}^n induces a metric and a pseudo-metric on $U(n)$ for $p \geq 1$. These metrics and pseudo-metric can be interpreted as “reasonable” cost functions described by tight QSL bounds involving $\langle |E|^p \rangle$ and $\mathcal{D}_p E$ respectively provided that $p \in [1, \pi/2]$. There is an open problem along this line of study. Our numerical study strongly suggests that $d_{p,\bar{\mu}}(U, V) = \sum_{j=1}^n \mu_j^\downarrow |\theta_j^\downarrow| [\theta_j^\downarrow (UV^{-1})]^p$ is a metric on $U(n)$ for $0 < p < 1$ whenever $\mu_1^\downarrow > 0$. It is instructive to prove this conjecture and to relate it to a tight QSL bound.

ACKNOWLEDGMENTS

We would like to thank C.-H. F. Fung for his valuable discussions. This work is supported under the RGC grant HKU 700709P of the HKSAR Government.

-
- [1] G. Wang and M. Ying, Phys. Rev. A **73**, 042301 (2006).
 - [2] A. Chefles, A. Kitagawa, M. Takeoka, M. Sasaki, and J. Twamley, J. Phys. A **40**, 10183 (2007).
 - [3] R. Duan, Y. Feng, and M. Ying, Phys. Rev. Lett. **98**, 100503 (2007), and the erratum in Phys. Rev. Lett. **98**, 129901(E) (2007).
 - [4] N. Margolus and L. B. Levitin, in *Proceedings of the 4th workshop on physics and computation (PHYSCOMP 96)*, edited by T. Toffoli, M. Biafore, and J. Leaõ (New England Complex Systems Institute, Cambridge, MA, 1996) p. 208.
 - [5] N. Margolus and L. B. Levitin, Physica D **120**, 188 (1998).
 - [6] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **67**, 052109 (2003).
 - [7] H. F. Chau, Phys. Rev. A **81**, 062133 (2010).
 - [8] K. Bhattacharyya, J. Phys. A **16**, 2993 (1983).
 - [9] A. Uhlmann, Phys. Lett. A **161**, 329 (1992).
 - [10] H. F. Chau, Quant. Inform. & Comp. **11**, 721 (2011).
 - [11] H. F. Chau, C.-K. Li, Y.-T. Poon, and N.-S. Sze, J. Phys. A (2012), accepted, arXiv:1107.1047v2.
 - [12] K.-Y. Lee, *Metrics of unitary matrices and their applications in quantum information theory*, Master's thesis, Univ. of Hong Kong, Hong Kong (2011).
 - [13] See, for example, A. N. Shiryaev, "Probability," (Springer, Berlin, 1995) p. 193, 2nd ed. for a proof.